Complete Intersections of Two Quadrics and Galois Cohomology

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Abstract

For each nonsingular hyperelliptic curve of arbitrary genus, we construct a natural injection from the Galois cohomology of 2-torsion subgroups of Jacobian varieties of the curve to the set of isomorphism classes of nonsingular complete intersections of two quadrics. This gives a generalization of the result of Flynn and Skorobogatov.

Introduction

Flynn and Skorobogatov constructed an injection from the Galois cohomology of 2-torsion subgroups of Jacobian varieties of nonsingular hyperelliptic curves of genus 2 to the isomorphism classes of del Pezzo surfaces of degree 4 (see [Flynn], [Skor2]). They also proved that images of the injections for all such curves cover the whole set of isomorphism classes of del Pezzo surfaces, and characterized the image of trivial elements. In this paper, we generalize these results for nonsingular hyperelliptic curves of arbitrary genus.

Let k be a field of characteristic not equal to 2 with separable closure k^s, and G_k the absolute Galois group $\mathrm{Gal}(k^s/k)$. A separable polynomial $f(x)$ of odd degree $n = 2m+1 \geq 3$ over k defines an étale k-algebra $L := k[x]/(f(x))$ of dimension $2m+1$ and a nonsingular projective hyperelliptic curve C over k of genus m whose affine model is given by

$$y^2 = f(x).$$

Let J_C be the Jacobian variety of C. The group of 2-torsion points $J_C[2]$ is isomorphic to $G := (\mathrm{Res}_{L/k}\,\mu_2)/\Delta(\mu_2)$ as G_k-modules, where $\mathrm{Res}_{L/k}$ denotes Weil restriction and

$$\Delta : \mu_2 \hookrightarrow \mathrm{Res}_{L/k}\,\mu_2$$

is the diagonal embedding. Since the short exact sequence

$$0 \longrightarrow \mu_2 \xrightarrow{\Delta} \mathrm{Res}_{L/k}\,\mu_2 \longrightarrow G = J_C[2] \longrightarrow 0$$

is split, we have isomorphisms

$$H^1(G_k, J_C[2]) \cong H^1(G_k, G) \cong L^\times/k^\times L^{\times 2}.$$

On the other hand, we know that elements of $H^1(G_k, J_C[2])$ correspond to isomorphism classes of 2-coverings of the Jacobian variety J_C.

The main goal of this paper is the following theorem:

Theorem 0.1. *Let $m \geq 1$ be a positive integer, $n = 2m+1$ an odd integer, and k a field of characteristic not equal to 2. Assume $\#k \geq n$.*
(i) Let C be a nonsingular projective hyperelliptic curve of genus m defined over k whose affine model is given by $y^2 = f(x)$ where $f(x)$ is a separable polynomial of degree $n = 2m+1$. There exists a natural injection

$$i_C \colon H^1(G_k, J_C[2]) \hookrightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of nonsingular complete} \\ \textit{intersections of two quadrics in } \mathbb{P}_k^{n-1} \textit{ over } k \end{array} \right\} =: \mathsf{ciq}_n$$

such that $i_C(0)$ is a variety containing a k-subvariety isomorphic to $\mathbb{P}_k^{m-1}$ (we call it quasi-split).
(ii) Conversely, for any isomorphism class $[X]$ of nonsingular complete intersection X of two quadrics in $\mathbb{P}_k^{n-1}$ over k, there exists a nonsingular projective hyperelliptic curve C of genus m and $\eta \in H^1(G_k, J_C[2])$ such that $i_C(\eta) = [X]$.

Recall that we have the following short exact sequence

$$0 \longrightarrow J_C(k)/2J_C(k) \longrightarrow H^1(G_k, J_C[2]) \longrightarrow H^1(G_k, J_C)[2] \longrightarrow 0.$$

However, we note that while the image of i_C depends only on the $\overline{k}$-isomorphism class of C and k-rational Weierstrass point $P = \infty$, the image of $J_C(k)/2J_C(k)$ inside ciq_n depends on the k-isomorphism class of C. So we need another data

In Section 1, we define ciq_n and other sets orb_n, gc_n, and hec_n. Then we construct explicit bijections between them. In Section 2, we define the map i_C using the bijections in Section 1, and discuss what objects are quasi-split in each set. In Section 3, we give examples and compare with Skorobogatov's result.

Notations

In this paper, we fix a field k of characteristic not equal to 2 and its algebraic closure $\overline{k}$. Denote by k^s the separable closure in $\overline{k}$, and $G_k = \mathrm{Gal}(k^s/k)$ the absolute Galois group. Let $m \geq 1$ be a positive integer and $n = 2m+1$. We use the following notation: V an n-dimensional k-vector space, $\{v_i\}_{0 \leq i \leq n-1}$ a basis of V, W a two-dimensional k-vector space, $\{w_i\}_{i=0,1}$ a basis of W. Also we use V^*, W^* for the dual spaces of V, W, and $\{v_i^*\}, \{w_i^*\}$ for the dual bases with respect to $\{v_i\}, \{w_i\}$. For $r \geq 2$, we denote by $\mathrm{Sym}_r V$ the r-ic symmetric subspace of $\bigotimes^r V$, and $\mathrm{Sym}^r V$ the r-ic symmetric quotient of $\bigotimes^r V$. We use similar notation for the alternating subspace $\wedge_r V$ and the alternating quotient space $\wedge^r V$. When we say L is a k-algebra of degree n, we mean L is a commutative k-algebra whose dimension is n as a k-vector space.

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1 Definition of $\mathsf{ciq}_n, \mathsf{orb}_n, \mathsf{gc}_n$, and hec_n

In this section, we fix an odd integer $n = 2m+1 \geq 3$. Assume that k contains at least n elements.

1.1 Definition and Relation of Geometric objects ciq_n and Orbital Objects orb_n

Before discussion, we recall some definitions from [Reid]. Let V be an n-dimensional k-vector space. Fix a two-dimensional subspace W of the space of quadratic forms $\mathrm{Sym}_2 V^*$, which we call a *pencil of quadratic forms*. Each quadratic form defines a quadric in $\mathbb{P}(V)$, so W defines a *pencil of quadrics*. We identify them and abuse the notation.

For a quadratic form $Q \in \mathrm{Sym}_2 V^*$, we define the *degeneracy* as $\dim_k \ker(Q : V \to V^*)$. Two pencils W, W' are *projectively equivalent* if there exists a k-linear isomorphism $V \xrightarrow{\sim} V$ which induces an isomorphism $W \xrightarrow{\sim} W'$.

Definition 1.1. (a) A pencil of quadratic forms $W \subset \mathrm{Sym}_2 V^*$ is *nonsingular* when it satisfies the following two conditions:
(i) the degeneracy of a quadratic form $Q \in W \otimes_k \overline{k}$ is at most one,
(ii) $\bigcap_{Q \in W \otimes_k \overline{k}} \ker(Q) = \{0\}$.
(b) The *determinant form* $\det \in \mathrm{Sym}^n W^* \otimes (\wedge^n V^*)^{\otimes 2}$ of a pencil of quadratic forms $W \subset \mathrm{Sym}_2 V^*$ is

$$W \ni w \mapsto \wedge^n w \in (\wedge^n V^*)^{\otimes 2}.$$

The following theorem from [Reid] says that two quadrics which intersect completely and smoothly can be identified with a basis of nonsingular pencil of quadratic forms.

Proposition 1.2. *[Reid, Proposition 2.1]*
Let V be an n-dimensional k-vector space, $W \subset \mathrm{Sym}_2 V^*$ a pencil of quadrics in $\mathbb{P}(V)$, and $X := \bigcap_{Q \in W \otimes_k \overline{k}} Q \subset \mathbb{P}(V \otimes_k \overline{k})$. Then the following conditions are equivalent:

(a) *X is nonsingular and codimension two in $\mathbb{P}(V)$ (this means X is a nonsingular complete intersection of two quadrics over k).*
(b) *The pencil W is nonsingular in the sense of Definition 1.1.*
(c) *The determinant form $\det \in \mathrm{Sym}^n W^* \otimes (\wedge^n V^*)^{\otimes 2}$ has nonzero discriminant.*

If one considers over $k = k^s$, these conditions are equivalent to another condition: for any basis $\{w_0, w_1\}$ of W, there exists a basis $\{v_i\}$ of V and $\lambda_i \in k^s (0 \leq i \leq n-1)$ such that for any $x_i \in k^s$ $(0 \leq i \leq n-1)$,

$$w_0\Big(\sum_{i=0}^{n-1} x_i v_i\Big) = \sum_{i=0}^{n-1} x_i^2$$
$$w_1\Big(\sum_{i=0}^{n-1} x_i v_i\Big) = \sum_{i=0}^{n-1} \lambda_i x_i^2,$$

and $\lambda_i \neq \lambda_j$ for $i \neq j$. Furthermore, the basis $\{v_i\}$ is unique up to change of signs in front of the v_i.

Definition 1.3. We define ciq_n as the set of isomorphism classes of nonsingular complete intersections of two quadrics over k in $\mathbb{P}(V)$.

Let X be a nonsingular complete intersection of two quadrics in $\mathbb{P}(V)$, and $-K_X$ its anticanonical divisor. Then we obtain

$$H^0(X, -K_X) \cong H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \cong V^*$$

and

$$\ker\left(H^0\left(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(2)\right) \to H^0(X, -2K_X)\right) \cong W \subset \mathrm{Sym}_2 V^*.$$

Let us take another n-dimensional k-vector space V' and a nonsingular complete intersection Y of two quadrics in $\mathbb{P}(V')$, which is isomorphic to X. Since $H^0(X, -K_X)$ is a complete linear system, an isomorphism $X \xrightarrow{\sim} Y$ induces an isomorphism $V \xrightarrow{\sim} V'$ and $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V')$. The following diagram is commutative:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(V) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ Y & \longrightarrow & \mathbb{P}(V') \end{array}.$$

Therefore, the isomorphism class of X determines a projective equivalence class of pencils of quadrics in $\mathbb{P}(V)$. By Proposition 1.2, the pencil of quadratic forms $W \subset \mathrm{Sym}_2 V^*$ defined by X is nonsingular.

Conversely, a nonsingular pencil of quadratic forms $W \subset \mathrm{Sym}_2 V^*$ defines a nonsingular complete intersection of two quadrics X by Proposition 1.2. Since two pencils W, W' are projectively equivalent if and only if a coordinate change of $\mathbb{P}(V)$ induces an isomorphism $W \xrightarrow{\sim} W'$, all elements in the projective equivalence class of W defines a variety isomorphic to X. These two maps are inverses of each other.

Furthermore, the set of projective equivalence classes of nonsingular pencils of quadratic forms is identified with the set of stable orbits of the quotient $k^2 \otimes \mathrm{Sym}_2 k^n / \mathrm{GL}_2(k) \times \mathrm{GL}_n(k)$, where a $\mathrm{GL}_2(k) \times \mathrm{GL}_n(k)$-orbit W is *stable* when its determinant form in $\mathrm{Sym}^n W^* \otimes (\wedge^n V^*)^{\otimes 2}$ has nonzero discriminant.

Definition 1.4. We define orb_n as the set of stable orbits of $k^2 \otimes \mathrm{Sym}_2 k^n / \mathrm{GL}_2(k) \times \mathrm{GL}_n(k)$.

Consequently, we construct the following bijections:

$$\mathsf{ciq}_n \overset{\sim}{\leftrightarrow} \left\{ \begin{array}{l} \text{projective equivalence classes} \\ \text{of nonsingular pencils of} \\ \text{quadrics in } \mathbb{P}_k^{n-1} \text{ over } k \end{array} \right\} \overset{\sim}{\leftrightarrow} \mathsf{orb}_n. \tag{1}$$

Next, we discuss about the characteristic scheme. Closed subschemes $S \subset \mathbb{P}(W)$ and $S' \subset \mathbb{P}(W')$ are *projectively equivalent* when there exists an isomorphism $\mathbb{P}(W) \xrightarrow{\sim} \mathbb{P}(W')$ which induces an isomorphism $S \xrightarrow{\sim} S'$.

Definition 1.5. The *characteristic scheme* of $(V, W) \in \mathsf{orb}_n$ is the subscheme of $\mathbb{P}(W)$ defined by $\det W$. The *characteristic scheme* of $X \in \mathsf{ciq}_n$ is the characteristic scheme of the corresponding class (V, W), where $V = H^0(X, -K_X)$ and $W = \ker\left(H^0\left(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(2)\right) \to H^0(X, -2K_X)\right)$.

By definition, the bijections (1) preserve the projective equivalence class of the characteristic schemes.

1.2 Definition of Algebraic Objects gc_n and their Relation to orb_n

In this subsection, we define the third set gc_n and discuss its relation to the set orb_n.

Definition 1.6. Let us consider the following triple:

- L is an étale k-algebra of degree n,
- $\theta \in L$ is a generator of L as a k-algebra,
- α is an element of $L^\times/k^\times L^{\times 2}$.

Two triples $(L,\theta,\alpha),(L',\theta',\alpha')$ are *equivalent* if there exists a k-algebra isomorphism $L \xrightarrow{\sim} L'$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(k)$ such that $\dfrac{a\theta+b}{c\theta+d} \mapsto \theta'$ and $\alpha \mapsto \alpha'$. We define gc_n as the set of equivalence classes of triples (L,θ,α) as above.

We note that $H^1(G_k, \mathrm{Res}_{L/k}(\mu_2)/\Delta(\mu_2)) \cong L^\times/k^\times L^{\times 2}$ because n is odd.

Now we shall construct a bijection between orb_n and gc_n. First, we construct a map $\mathsf{orb}_n \to \mathsf{gc}_n$. In fact, this step is the hardest part of the construction of i_C. Take an element $(V,W) \in \mathsf{orb}_n$ and fix a basis $\{w_0,w_1\}$ of W so that w_0 is a nondegenerate quadratic form on V (in this choice, we use the assumption $\#k \geq n$). Consider W as a two-dimensional k-linear subspace of $V^* \otimes V^* \xrightarrow{\sim} \mathrm{Hom}_k(V,V^*)$ and put

$$w_0^{-1}W := \{w_0^{-1} \circ w \mid w \in W\} \subset \mathrm{End}\, V.$$

The space $w_0^{-1}W$ is a two-dimensional k-vector space, and $\{\mathrm{id} = w_0^{-1} \circ w_0, \theta := w_0^{-1} \circ w_1\}$ is a basis of $w_0^{-1}W$.

Lemma 1.7. *The k-subalgebra L_W of $\mathrm{End}\, V$ generated by $w_0^{-1}W$ is an étale k-algebra of degree n. Moreover, V is a free L_W-module of rank one.*

Proof. Write the characteristic polynomial of θ as $P_\theta(x)$. By definition,

$$P_\theta(x) = \det(w_0)^{-1} \circ \det(xw_0 - w_1).$$

Since $\det \in \mathrm{Sym}^n W^* \otimes (\wedge^n V^*)^{\otimes 2}$ has nonzero discriminant by Proposition 1.2, we see $L_W = k[x]/(P_\theta(x))$ and it is an étale k-algebra of degree n.

Then L_W can be written as a product of fields $\prod_{s=1}^r K_s$, where K_s is a finite separable extension of k. There exist idempotent elements $e_s \in L_W$ such that $L_W e_s = K_s$ and $\sum_{s=1}^r e_s = 1_L$, and vectors $v'_s \in V$ such that $e_s v'_s \neq 0$. Then put $v' := \sum_{s=1}^r e_s v'_s$.

If one takes an element $\ell \in L_W$ such that $\ell v' = 0$, we find $\ell e_s v' = \ell e_s v'_s = 0$ for all s. Then $\ell e_s = 0$. Since $\ell = \sum_{s=1}^r \ell e_s$, we have $\ell = 0$. By comparing dimensions as k-vector spaces, we find v' defines a k-isomorphism $L_W \xrightarrow{\sim} V$. Hence V is a free L_W-module of rank one. $\square$

From Lemma 1.7, we fix an L_W-module isomorphism $\gamma\colon L_W \xrightarrow{\sim} V$. Then for each $w \in W$, we can define $\widetilde{w} = (\gamma^* \otimes \gamma^*)(w) \in \mathrm{Sym}_2 L_W^*$. Put $\widetilde{W}$ for the two-dimensional k-subspace of $\mathrm{Sym}_2 L_W^*$ consisting of such elements.

Lemma 1.8. *All k-bilinear forms $\widetilde{w} \in \widetilde{W} \subset \mathrm{Sym}_2 L_W^*$ satisfy*

$$\widetilde{w}(x\theta, y) = \widetilde{w}(x, \theta y) \quad (\text{for } x, y \in L_W) \tag{2}$$

Proof. Since γ is an isomorphism as L_W-module, $\gamma(x\theta) = \gamma(x)\theta$. By regarding $\mathrm{Sym}_2 V^* \subset V^* \otimes V^*$ as the k-subspace of $\mathrm{Hom}_k(V,V^*)$, we calculate

$$\begin{aligned}\widetilde{w}(x\theta, y) &= w(\gamma(x)\theta, \gamma(y)) \\ &= w(\gamma(x)\theta)(\gamma(y)).\end{aligned}$$

Because $\theta = w_0^{-1} \circ w_1$ and we can write $w = bw_0 + aw_1$, we obtain

$$w(\gamma(x)\theta)(\gamma(y)) = (bw_1 + aw_1 \circ w_0^{-1} \circ w_1)(\gamma(x))(w_0^{-1} \circ w(\gamma(y))).$$

Now this lemma follows by the symmetry of w_0 and w_1. $\square$

Let $\mathcal{W}$ be the subspace consisting of $\widetilde{w} \in \mathrm{Sym}_2 L_W^*$ satisfying (2). For $\phi \in L_W^*$, this space $\mathcal{W}$ contains $\phi(x, y) := \phi(xy)$. The next lemma shows that all elements in $\mathcal{W}$ come from some element of L_W^*.

Lemma 1.9. *$\mathcal{W} = \widetilde{w_0 L_W} = \{(\gamma \otimes \gamma)(w_0 \circ (\times\alpha \otimes id_V)) \mid \alpha \in L_W \subset \mathrm{End}\, V\}$. In particular, by the action $(\alpha \cdot \widetilde{w})(x, y) = \widetilde{w}(x\alpha, y)$, $\mathcal{W}$ is a free L_W-module of rank one.*

Proof. Since L_W is an étale k-algebra, there are n different k-algebra homomorphisms $\iota_i \colon L_W \to k^s (0 \le i \le n-1)$. We put $\theta_i := \iota_i(\theta)$. By Proposition 1.2, all θ_i are distinct, and we can choose a k^s-basis $\{e_i\}$ of $L_W \otimes k^s$ such that for any $x_i \in k^s (i = 0 \le i \le n-1)$,

$$w_0(\sum_{i=0}^{n-1} x_i e_i) = \sum_{i=0}^{n-1} x_i^2,$$
$$w_1(\sum_{i=0}^{n-1} x_i e_i) = \sum_{i=0}^{n-1} \theta_i x_i^2.$$

Then we can check $\theta(x_i e_i) = (w_0^{-1} \circ w_1)(x_i e_i) = \theta_i x_i e_i \in L_W$, and $L_W \otimes_k k^s$ coincides with the subspace of all elements in $\mathrm{End}(L_W \otimes k^s)$ having diagonal matrix representation.

Since all θ_i are distinct, the condition (2) says all $q \in \mathcal{W} \otimes_k k^s$ have diagonal matrix representation with respect to the basis $\{e_i\}$. Such quadratic forms are in $\widetilde{w_0 L_W} \otimes_k k^s$. $\square$

To summarize, we obtain the following diagram:

$$\begin{tikzcd}
 & & & \mathrm{End}\, L_W \arrow[d, "\cong"] & \\
 & \mathrm{Sym}_2 L_W^* \arrow[r, leftrightarrow, "\overset{\gamma}{\cong}"] & \mathrm{Sym}_2 V^* \arrow[r, hook, "w_0^{-1}"] & \mathrm{End}\, V & \\
 & \cup & \cup & \cup & \\
L\text{-module} & \mathcal{W} = \widetilde{w_0 L_W} \arrow[r, leftrightarrow] & w_0 L_W \arrow[r, leftrightarrow] & L_W \arrow[r, leftrightarrow, "\overset{\gamma}{\cong}"] & V \\
 & \cup & \cup & \cup & \\
\text{2-dim. sp.} & \widetilde{W} \arrow[r, leftrightarrow] & W \arrow[r, leftrightarrow] & w_0^{-1} W & \\
 & \cup & \cup & \cup & \\
\text{basis} & \{\widetilde{w}_0, \widetilde{w}_1\} \arrow[r, leftrightarrow] & \{w_0, w_1\} \arrow[r, leftrightarrow] & \{1, \theta\} &
\end{tikzcd}$$

As the final step of the construction of a map $\mathsf{orb}_n \to \mathsf{gc}_n$, we shall define $\alpha \in L_W^\times / k^\times L_W^{\times 2}$. In our condition, a k-algebra L_W has the canonical k-basis $\{\theta^i\}_{0 \le i \le n-1}$. Let $\{\check{\theta}_i\}_{0 \le i \le n-1}$ be the dual basis of $\{\theta^i\}$, and put $t_\theta(x, y) := \check{\theta}_{n-1}(xy)$. It is a nondegenerate k-bilinear form on L_W because the matrix representation of t_θ with respect to the basis $\{\theta^i\}$ is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & * \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & * & * \\ 1 & * & \cdots & * & * \end{pmatrix},$$

and by definition, $t_\theta \in \mathcal{W}$. By Lemma 1.9, for each $\widetilde{w} \in \mathcal{W}$, there exists a unique element $\alpha_{w,\theta}$ such that $\widetilde{w} = \alpha_{w,\theta} \cdot t_\theta$. Since $\widetilde{w_0}$ is nondegenerate, we obtain $\alpha \in L_W^\times$ satisfying $\widetilde{w_0} = \alpha \cdot t_\theta$. This α is what we need.

On above construction of the map $\mathsf{orb}_n \to \mathsf{gc}_n$, we fix two data: (i) a k-frame (i.e. ordered basis) $\{w_0, w_1\}$ of W, and (ii) an isomorphism of L_W-modules, $\gamma\colon L_W \xrightarrow{\sim} V$. If we change γ to $\gamma' = c\gamma$ for some $c \in L_W^\times$,

$$w(\gamma'(x), \gamma'(y)) = w(\gamma(x), c^2\gamma(y)). \tag{3}$$

Hence α is multiplied by an element of $L_W^{\times 2}$. To see the effects of change of frame, we need another lemma. Recall the trace form of an étale algebra is nondegenerate.

Lemma 1.10. *[Ser, III.6, Lemma 2]*
Let $P_\theta(x) := \mathrm{N}_{L_W/k}(x - \theta) \in k[x]$ be the characteristic polynomial of θ. Then

$$\mathrm{Tr}_{L_W/k}\left(\frac{x}{P_\theta'(\theta)}\right) = \check{\theta}_{n-1}(x).$$

Corollary 1.11. *Let θ' be another generator of L_W as a k-algebra. Then we obtain*

$$t_{\theta'} = \frac{P_\theta'(\theta)}{P_{\theta'}'(\theta')} \cdot t_\theta.$$

Proof.

$$\begin{aligned} t_\theta'(x) &= \mathrm{Tr}_{L_W/k}\left(\frac{x}{P_{\theta'}'(\theta')}\right) \\ &= \mathrm{Tr}_{L_W/k}\left(\frac{P_\theta'(\theta)}{P_{\theta'}'(\theta')}\frac{x}{P_\theta'(\theta)}\right). \quad \square \end{aligned}$$

If the frame of W changes to $\{dw_0 + cw_1, bw_0 + aw_1\}$ (and assume that $dw_0 + cw_1$ is nondegenerate), we can see that θ changes to $\theta' := \frac{a\theta+b}{c\theta+d}$. Moreover, we can calculate

$$\frac{P_\theta'(\theta)}{P_{\theta'}'(\theta')} = \mathrm{N}_{L/k}(c + d\theta)\left(\frac{c + d\theta}{ad - bc}\right)^{n-1}. \tag{4}$$

Because n is odd and by (3), α is well-defined up to $k^\times L_W^{\times 2}$. Now the well-definedness of our map $\mathsf{orb}_n \to \mathsf{gc}_n$ can be checked easily.

The construction of the inverse map $\mathsf{gc}_n \to \mathsf{orb}_n$ is easy; we send $(L, \theta, \alpha) \mapsto (L, k\alpha \cdot t_\theta + k\alpha\theta \cdot t_\theta)$. A change $\alpha \mapsto a\alpha\,(a \in k^\times)$ does not affect the k-vector spaces L and $k\alpha t_\theta + k\alpha\theta t_\theta$. When one changes $\alpha \mapsto c^2\alpha\,(c \in L^\times)$, the isomorphism as L-module $c\colon L \xrightarrow{\sim} L; x \mapsto cx$ absorbs its effects. An argument similar to (4) shows a change $\theta \mapsto \dfrac{a\theta + b}{c\theta + d}$ does not affect. Hence this map is well-defined. Note that $(L, \theta, 1_L)$ is mapped to $(L, kt_\theta + k\theta t_\theta)$ (see Section 2).

We have to check these maps are inverses of each other. First, we compare (V, W) and $(L, k\alpha \cdot t_\theta + k\alpha\theta \cdot t_\theta)$ obtained by the composition map $\mathsf{orb}_n \to \mathsf{gc}_n \to \mathsf{orb}_n$. But an isomorphism as L-module $\gamma : L \xrightarrow{\sim} V$ gives an isomorphism between (V, W) and $(L, k\alpha \cdot t_\theta + k\alpha\theta \cdot t_\theta)$. Conversely, we take $(L, \theta, \alpha) \in \mathsf{gc}_n$. Because $(\alpha \cdot t_\theta)^{-1} \circ (\alpha\theta \cdot t_\theta) = (\gamma^* \otimes$

$\gamma)^{-1}(\theta) \in \operatorname{End} V$ and $\gamma^* \otimes \gamma\colon \operatorname{End} V \xrightarrow{\sim} \operatorname{End} L$, we recover L and the class of θ. Since $(t_\theta)^{-1} \circ (\alpha \cdot t_\theta) = (\gamma^* \otimes \gamma)^{-1}(\alpha) \in \operatorname{End} V$, we recover $\alpha \in L^\times$ up to $k^\times L^{\times 2}$.

In conclusion, we constructed the following bijection:

$$\mathsf{orb}_n \overset{\sim}{\leftrightarrow} \mathsf{gc}_n = \left\{ \text{ equivalence classes of } (L, \theta, \alpha) \right\}. \tag{5}$$

Finally, we discuss the characteristic schemes.

Definition 1.12. The *characteristic scheme* of a triple $(L, \theta, \alpha) \in \mathsf{gc}_n$ is the subscheme of $\mathbb{P}^1_k$ defined by $P_\theta(x)$.

The isomorphism between the characteristic schemes of orb_n and gc_n is given by w_0. The bijection $\mathsf{orb}_n \xrightarrow{\sim} \mathsf{gc}_n$ preserves the projective equivalence classes of characteristic schemes.

1.3 Definition of hec_n and its Relation to gc_n

In this subsection, we define a set hec_n and construct a bijection between gc_n and hec_n preserving characteristic schemes. Let C be a nonsingular projective hyperelliptic curve over k of genus m, and $\iota_C\colon C \to C$ the hyperelliptic involution of C. Let T_C be the subscheme of $\mathbb{P}^1_k$ where $C \to C/\iota_C \cong \mathbb{P}^1_k$ ramifies. Because n is odd, T_C is a $(n+1)$ points subscheme of $\mathbb{P}^1_k$. Let $P \in T_C(k)$ be a k-rational point, and put $S_C := T_C \setminus \{P\}$. Then a k-algebra $L_C := H^0(S_C, \mathcal{O}_{S_C})$ is an étale k-algebra of degree n, and the following proposition holds.

Proposition 1.13. *(c.f. [Scha, Theorem1.1]) Let J_C be the Jacobian variety of C, and $J_C[2]$ the group subvariety of 2-torsion points on J_C. Then there exist isomorphisms of groups:*

$$H^1(G_k, J_C[2]) \cong H^1\left(G_k, \operatorname{Res}_{L_C/k}(\mu_2)/\Delta(\mu_2)\right) \cong L_C^\times/k^\times L_C^{\times 2}.$$

So the projective equivalence class of the n points subscheme $S_C \subset \mathbb{P}^1_k$ determines the Galois cohomology group $H^1(G_k, J_C[2])$ as an abstract group. But this isomorphism depends on the choice of P, which corresponds to the choice of generator $\theta \in L_C$.

Definition 1.14. Let us consider the following triple:

- C is a nonsingular projective hyperelliptic curve over k of genus m,
- $P \in T_C(k)$ is a image of a k-rational Weierstrass point of C,
- η is an element of $H^1(G_k, J_C[2])$.

Two such triples $(C, P, \eta), (C', P', \eta')$ are *equivalent* if the following conditions hold:
(i) S_C and $S_{C'}$ are projectively equivalent (this means there is an isomorphism $\Phi\colon \mathbb{P}^1_k \xrightarrow{\sim} \mathbb{P}^1_k$ which induces an isomorphism $\Phi|_{S_C}\colon S_C \xrightarrow{\sim} S_{C'}$),
(ii) η is mapped to η' through the following isomorphism (see Proposition 1.13) :

$$H^1(G_k, J_C[2]) \xrightarrow{\sim} L_C^\times/k^\times L_C^{\times 2} \overset{\phi}{\xrightarrow{\sim}} L_{C'}^\times/k^\times L_{C'}^{\times 2} \xrightarrow{\sim} H^1(G_k, J_{C'}[2])$$

where ϕ is induced by $\Phi|_{S_C}$.

We define hec_n as the set of equivalence classes of above triples. The *characteristic scheme* of the triple (C, P, η) is $S_C \subset \mathbb{P}^1_k$.

Remark 1.15. Note that two triples $(C, P, \eta), (C', P', \eta')$ can be equivalent in the sense of Definition 1.14 even if C, C' are not $\overline{k}$-isomorphic.

Let us construct a bijection between gc_n and hec_n preserving characteristic schemes.

For $(L, \theta, \alpha) \in \mathsf{gc}_n$, let $P_\theta(x)$ be the characteristic polynomial of θ, $S = \operatorname{Spec} k[x]/(P_\theta(x))$ the closed subscheme of $\mathbb{P}^1_k$ defined by $P_\theta(x)$, and $T := S \cup \{\infty\}$. Take an arbitrary non-singular hyperelliptic curve C satisfying $T_C = T$, $P = \infty$, (i.e. whose affine model is $y^2 = aP_\theta(x)$ for some $a \in k^\times$) and η is given by the image of α by the isomorphisms of Proposition 1.13.

If (L, θ, α) is equivalent to (L', θ', α'), there exists a k-algebra isomorphism $f\colon L \xrightarrow{\sim} L'$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$, such that

$$f\left(\frac{a\theta + b}{c\theta + d}\right) = \theta'.$$

It induces an isomorphism $T \xrightarrow{\sim} T', \infty \mapsto [a : c] \in \mathbb{P}^1_k, \alpha \mapsto \alpha'$, hence $(C, P, \eta), (C', P', \eta')$ are equivalent. So the induced map $\mathsf{gc}_n \to \mathsf{hec}_n$ is well-defined, and preserves the projective equivalence classes of characteristic schemes.

On the other hand, for an element $(C, P, \eta) \in \mathsf{hec}_n$, take its representative and put $L_C = H^0(S_C, \mathcal{O}_{S_C})$. To take θ, consider the restriction morphism of sheaves, $\varphi\colon \mathcal{O}_{\mathbb{P}^1_k}(1) \to \mathcal{O}_{S_C}(1)$. Let us choose $x \in H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1))$ having a simple pole at P, and $\theta := \varphi(x) \in H^0(S_C, \mathcal{O}_{S_C}(1))$. Under the isomorphism $\mathbb{P}^1_k \backslash \{P\} \cong \operatorname{Spec} k[x]$, we see that $L_C \cong k[x]/(P_\theta(x))$, and θ is the image of x. Cohomological element α is given by the isomorphisms of Proposition 1.13.

Assume that two triples $(C, P, \eta), (C', P', \eta')$ are equivalent. We obtain two triples $(L, \theta, \alpha), (L', \theta', \alpha')$. Since the characteristic schemes $S_C, S_{C'}$ are projectively equivalent, there is an automorphism of k-schemes $\mathbb{P}^1_k \xrightarrow{\sim} \mathbb{P}^1_k$ which induces $S_C \xrightarrow{\sim} S_{C'}$. It induces an isomorphism $L \xrightarrow{\sim} L'$. Since an automorphism of $\mathbb{P}^1_k$ is written as a fractional linear transformation, we can write

$$f(x) = \frac{ax + b}{cx + d} \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)\right).$$

Therefore, we have $\theta' = \dfrac{a\theta + b}{c\theta + d}$ and $\eta \mapsto \eta'$. Hence $(L, \theta, \alpha), (L', \theta', \alpha')$ are equivalent. The map $\mathsf{hec}_n \to \mathsf{gc}_n$ is well-defined, and preserves the projective equivalence classes of characteristic schemes.

We must check that the maps $\mathsf{gc}_n \to \mathsf{hec}_n$ and $\mathsf{hec}_n \to \mathsf{gc}_n$ are inverses of each other. Assume that (C, P, η) maps to (C', P', η') via the composition map of $\mathsf{hec}_n \to \mathsf{gc}_n \to \mathsf{hec}_n$. Since the maps preserve the projective equivalence classes of characteristic schemes, we may assume $P = P' = \infty$, and C(resp. C') have affine models $y^2 = f(x)$ (resp. $y^2 = af(x)$ for some $a \in k^\times$). Then the composition morphism

$$H^1(G_k, J_C[2]) \xrightarrow{\sim} L_C^\times/k^\times L_C^{\times 2} \overset{\phi=\mathrm{id}}{\xrightarrow{\sim}} L_C^\times/k^\times L_C^{\times 2} \xrightarrow{\sim} H^1(G_k, J_C[2]),$$

is identity, so $\eta = \eta'$.

On the other hand, take $(L, \theta, \alpha) \in \mathsf{gc}_n$ and put $(L, \theta, \alpha) \mapsto (C, \infty, \eta) \mapsto (L', \theta', \alpha)$. By construction of maps, we obtain an isomorphism of k-algebras $L \xrightarrow{\sim} L'$. We may assume

that the isomorphism maps θ to θ', and an analogous argument as above shows they are equivalent.

In conclusion, we constructed a bijection:

$$\mathsf{gc}_n = \{\text{equivalence classes of } (L,\theta,\alpha)\} \overset{\sim}{\leftrightarrow} \{\text{equivalence classes of } (C,P,\eta)\} = \mathsf{hec}_n \quad (6)$$

preserving the projective equivalence classes of characteristic schemes.

2 Definition of map i_C and Proof of Main Theorem

Now we define the map i_C in our main theorem (Theorem 0.1). Let C be a nonsingular hyperelliptic curve over k which has an affine model $y^2 = f(x)$ where $f(x)$ is a separable polynomial of degree n. Let $P = \infty \in \mathbb{P}^1_k$ be the point at infinity. The double covering $C \to \mathbb{P}^1_k$ is ramified at $\{\infty\} \cup \operatorname{Spec} k[x]/(f(x)) \subset \mathbb{P}^1_k$. Put $S_C := \operatorname{Spec} k[x]/(f(x))$. Then for $\eta \in H^1(G_k, J_C[2])$, let $i_C(\eta)$ be the element in ciq_n corresponding to $(C,P,\eta) \in \mathsf{hec}_n$. Recall that the isomorphism in Proposition 1.13 depends on the choice of P, or θ.

Proposition 2.1. *The map i_C is injective. Moreover, an element $[X] \in \mathsf{ciq}_n$ whose characteristic scheme is projectively equivalent to S_C is contained in image of i_C.*

Proof. Since our bijections preserve the projective equivalence classes of characteristic schemes, the second statement follows. For the first statement, it suffices to show that

$$H^1(G_k, J_C[2]) \to \mathsf{hec}_n \quad ; \quad \eta \mapsto (C,P,\eta)$$

is injective.

If (C,P,η) is equivalent to (C,P,η'), η maps to η' through the following isomorphism:

$$H^1(G_k, J_C[2]) \overset{\sim}{\to} L_C^\times/k^\times L_C^{\times 2} \overset{\phi=\mathrm{id}}{\overset{\sim}{\to}} L_C^\times/k^\times L_C^{\times 2} \overset{\sim}{\to} H^1(G_k, J_C[2]),$$

but this isomorphism is clearly identity map. $\square$

The rest of this section will study about special class of each sets.

Definition 2.2. A triple $(C,P,\eta) \in \mathsf{hec}_n$ is *quasi-split* if $\eta = 0$. An element of $\mathsf{gc}_n, \mathsf{orb}_n, \mathsf{ciq}_n$ is *quasi-split* if the corresponding element in hec_n is quasi-split.

A triple $(L,\theta,\alpha) \in \mathsf{gc}_n$ is quasi-split if and only if $\alpha = 1_L$ (see Proposition 1.13).

Which elements of $\mathsf{orb}_n, \mathsf{ciq}_n$ correspond to quasi-split objects? By the bijection we constructed, $(L,\theta,1_L) \in \mathsf{gc}_n$ maps to $(L, kt_\theta + k\theta t_\theta) \in \mathsf{orb}_n$. By Lemma 1.10 and the definition of t_θ, the pencil of quadratic forms has a common isotropic m-dimensional space

$$\mathcal{L}^{m-1} := \{a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1} \mid a_i \in k \quad (0 \le i \le m-1)\}.$$

This property characterizes the quasi-split objects in orb_n. In fact, if $w \in \mathrm{Sym}_2 L^*$ satisfies $w(\mathcal{L}^{m-1}) = w(\theta \cdot \mathcal{L}^{m-1}) = 0$, then the k-linear form $w_0(x) = w(x+1_L) - w(x) - w(1_L)$ of L kills $\mathcal{L}^{2m} = \mathcal{L}^{n-1}$. Hence $w_0 \in kt_\theta$. To summarize,

Theorem 2.3. *A pair $(V,W) \in \mathsf{orb}_n$ is quasi-split if and only if W has a common isotropic m-dimensional k-subspace. A nonsingular complete intersection of two quadrics $X \in \mathsf{ciq}_n$ is quasi-split if and only if X contains a linear k-subvariety isomorphic to $\mathbb{P}^{m-1}_k$.*

Corollary 2.4. *Let us fix a reduced n points subscheme S of $\mathbb{P}^1_k$. There is, up to isomorphism, a unique nonsingular complete intersection of two quadrics X in $\mathbb{P}^{2m}_k$ satisfying*

- *the characteristic scheme X is projectively equivalent to S,*
- *X contains a linear k-subvariety isomorphic to $\mathbb{P}^{m-1}_k$.*

Proof. By Theorem 2.3, such a variety X exists. So we only have to show the uniqueness. By Proposition 2.1, all such $[X]$ is in the image of i_C, and uniqueness follows from injectivity of i_C. □

These theorem and corollary finish our main Theorem 0.1, because quasi-split objects in ciq_n can be written as $i_C(0)$.

Remark 2.5. In [BG], Bhargava and Gross call quasi-split objects in orb_n *distinguished orbits*. And in [Skor2], Skorobogatov calls quasi-split objects in gc_5 *quasi-split del Pezzo surfaces* (see next section).

3 Examples

3.1 Case for $n = 3$: Four Points Subschemes of $\mathbb{P}^2$

This case is well-known. The nonsingular complete intersections of two quadrics in $\mathbb{P}^2$ are four points subschemes of $\mathbb{P}^2$ in general position (i.e. no three points are geometrically collinear). From Proposition 1.2, we can check the converse is true. In fact, if one takes a four points k-subscheme of $\mathbb{P}^2_k$ in general position, we can regard four points are $[1 : \pm 1 : \pm 1]$ in some coordinate system over k^s. So quadrics through such four points consist a two-dimensional subspace of $\mathrm{Sym}_2 k^3$. This shows four points subschemes of $\mathbb{P}^2_k$ in general position are nonsingular complete intersections of two quadrics in $\mathbb{P}^2_k$. Hence we obtain:

Proposition 3.1. *There is a bijection between isomorphism classes of four points k-subschemes of $\mathbb{P}^2_k$ in general position and projectively equivalence classes of nonsingular pencils in* $\mathrm{Sym}_2 k^3$*.*

A four points subscheme of $\mathbb{P}^2_k$ is *quasi-split* if and only if it contains a k-rational point.

3.2 Case for $n = 5$: del Pezzo Surfaces of Degree Four

We recall the definition of del Pezzo surfaces and some properties of them. For more details, see [Dolg] for example.

A *del Pezzo surface* over k is a nonsingular projective surface X over k with ample anticanonical divisor $-K_X$. Its *degree* is the self-intersection number $(-K_X)^2$. From now on, we concentrate on the case of degree four.

Let X be a del Pezzo surface over k of degree four. We know that $V = H^0(X, -K_X)$ is a five-dimensional k-vector space, and X is realized as a nonsingular complete intersection of two quadrics in $\mathbb{P}(V) \cong \mathbb{P}^4_k$. Conversely, all pairs of two quadrics in $\mathbb{P}^4_k$ which intersect completely and smoothly define del Pezzo surfaces of degree four.

An isomorphism of del Pezzo surfaces $X \xrightarrow{\sim} Y$ induces an isomorphism of k-vector spaces $H^0(X, -K_X) \xrightarrow{\sim} H^0(Y, -K_Y)$. Conversely, since $H^0(X, -K_X)$ is a complete linear

system, all isomorphisms of del Pezzo surfaces of degree four are induced by such k-linear isomorphisms.

Furthermore, the kernel of the map

$$\cdot|_X : \mathrm{Sym}_2(H^0(X, -K_X)) \to H^0(X, -2K_X)$$

is a two-dimensional k-vector space [*ibidem*]. Hence the isomorphism class of a del Pezzo surface of degree four X determines a projective equivalence class of pencil of quadrics Φ, and a projective equivalence class of pencil Φ determines the isomorphism class of X. This shows ciq_5 coincides with the set of isomorphism classes of del Pezzo surfaces of degree four over k. So we conclude:

Proposition 3.2. *There is a bijection between isomorphism classes of del Pezzo surfaces of degree four over k and projectively equivalence classes of nonsingular pencils of quadratic forms in* $\mathrm{Sym}_2 k^5$.

A del Pezzo surface of degree four X is called *quasi-split* when it contains a k-line. In [Skor2], Skorobogatov showed that a del Pezzo surface X defined over k is quasi-split if and only if X is the blow-up of $\mathbb{P}^2_k$ at a five points subscheme defined over k.

Now let us prove the compatibility of our bijection and Skorobogatov's bijection in [*ibidem*, Section 2]. To prove it, let us fix a reduced 5 points subscheme S of $\mathbb{P}^1_k$ and put $L := H^0(S, \mathcal{O}_S)$. First, the characteristic schemes of del Pezzo surfaces of degree four X coincides with S_X in [*ibidem*]. By Corollary 2.4, there is a unique isomorphism class of quasi-split del Pezzo surface of degree four over k whose characteristic scheme is projectively equivalent to S. So del Pezzo surfaces is quasi-split if and only if quasi-split as an element of in ciq_n. This gives another proof of the uniqueness of [*ibidem*, Theorem 2.3(b)].

Furthermore, let $(\mathsf{ciq}_n)\,|_S = i_C(L^\times/k^\times L^{\times 2})$ be the subset of ciq_n whose characteristic scheme is projectively equivalent to S. Then there is a canonical simply transitive action of $L^\times/k^\times L^{\times 2}$ on $(\mathsf{ciq}_n)\,|_S$. Explicitly, when we put $X = \{\alpha \cdot t_\theta(x,x) = \alpha\theta \cdot t_\theta(x,x) = 0\}$ and $\lambda \in L^\times/k^\times L^{\times 2}$, we can write

$$\lambda \cdot X = \{\alpha\lambda \cdot t_\theta(x,x) = \alpha\lambda\theta \cdot t_\theta(x,x) = 0\}\,.$$

This action coincides with the action in [*ibidem*], so our bijection and Skorobogatov's bijection are compatible.

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